Useful Equations:

$$\vec{v}=\frac{d\vec{x}}{dt},\quad \vec{a}=\frac{d\vec{v}}{dt}=\frac{d^2\vec{x}}{dt^2},\quad adx=vdv,\quad v=v_0+a_c t,\quad x=x_0+v_0t+\tfrac{1}{2}a_ct^2,\quad v^2={v_0}^2+2a_c\left(x-x_0\right)$$

Problem #1

From approximately what floor of a building must a box be dropped from an at-rest position so that it reaches a speed 55 mph when it hits the ground? Each floor is 12 ft higher than the one below it.

Start with $a_c = g = 32.2$ ft/s^2, 55 mph = 80.6667 ft/s, and use $v^2={v_0}^2+2a_c\left(x-x_0\right)$

$$80.6667^2=0^2+2\cdot 32.2\left(x-0\right)\Rightarrow x=\frac{80.6667^2}{64.4}=101.042\text{ ft}$$

(101.042 ft) / (12 ft)/floor $\boxed{= 8.4202 \text{ floors}}$

Problem #2

A particle is moving along a straight line such that when it is at the origin it has a velocity 4 m/s. If it begins to decelerate at the rate $a = -1.5\sqrt{v}$, determine the particle's position and velocity when $t = 2$ seconds.

An expression that relates a, v, and t would be handy. Use $a=\frac{dv}{dt}\Rightarrow -1.5\sqrt{v}=\frac{dv}{dt}$

Separate variables: $-1.5dt=\frac{dv}{\sqrt{v}}$ and integrate: $-1.5\int_{t=0}^{t}dt=\int_{v=4\text{ m/s}}^{v}v^{-\frac{1}{2}}dv$

$$-1.5t=2\left(\sqrt{v}-2\right)\Rightarrow v=\left(2-\tfrac{1.5}{2}t\right)^2 \text{ and at } t=2\text{ s } v=\left(2-\tfrac{1.5}{2}2\right)^2=\left(0.5\right)^2\boxed{=0.25\text{ m/s}}$$

For position, use $v=\frac{dx}{dt}$ with previous result $v=\left(2-\tfrac{1.5}{2}t\right)^2$, separate variables:

$$dx=\left(2-\tfrac{1.5}{2}t\right)^2dt\text{ , and integrate: }\int_{x=0}^{x}dx=\int_{t=0}^{t=2\text{ s}}\left(2-\tfrac{1.5}{2}t\right)^2dt=\int_{t=0}^{t=2\text{ s}}\left(0.5625t^2-3t+4\right)dt$$

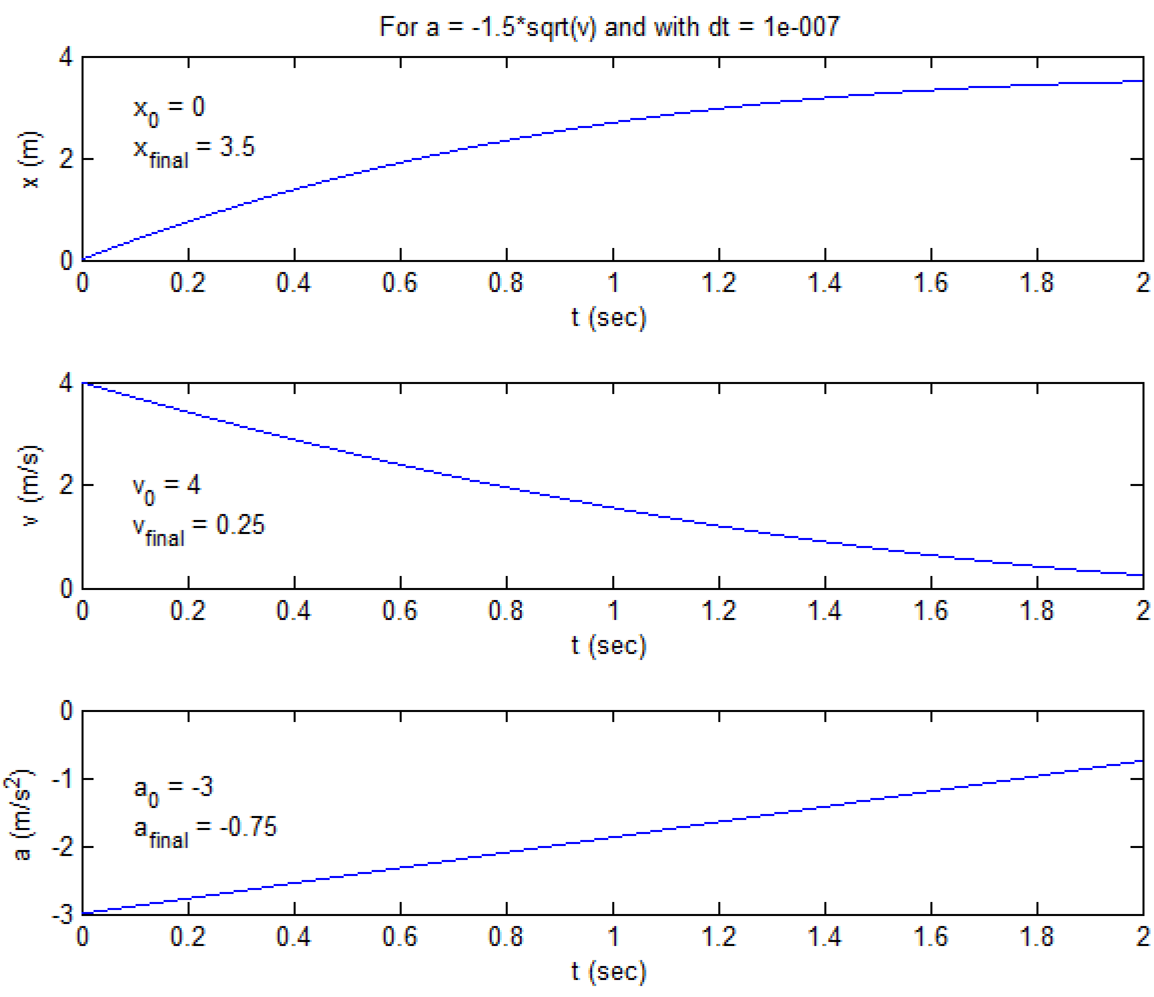
$$\tfrac{1}{3}0.5625t^3-\tfrac{1}{2}3t^2+4t\Big|_{t=0}^{t=2\text{ s}}=\tfrac{8}{3}0.5625-6+8\boxed{=3.5\text{ m}}$$

Alternatively, for position, started with $a=v\frac{dv}{dx}=-1.5\sqrt{v}\Rightarrow -1.5dx=\frac{v}{\sqrt{v}}dv\Rightarrow$

$$-1.5\int_0^x dx=\int_{v=4\text{ m/s}}^{v=0.25\text{ m/s}}v^{\frac{1}{2}}dv\Rightarrow -1.5x=\tfrac{2}{3}v^{\frac{3}{2}}\Big|_{v=4\text{ m/s}}^{v=0.25\text{ m/s}}\Rightarrow x=\frac{\tfrac{2}{3}\left(0.25\right)^{\frac{3}{2}}-\tfrac{2}{3}\left(4\right)^{\frac{3}{2}}}{-1.5}\Rightarrow x\boxed{=3.5\text{ m}}$$

Can also look at this problem numerically, in MATLAB for example, for an independent confirmation and to see how position, velocity, and acceleration vary with time.

```
clear all; clc; format compact; tic
t0 = 0; tn = 2; dt = 0.0000001; % setup time
t = linspace(t0, tn, (tn-t0)/dt + 1); % define all time steps
x = zeros(size(t)); v = zeros(size(t)); a = zeros(size(t)); % i 4 speed
x(1) = 0; v(1) = 4; a(1) = -1.5*sqrt(v(1)); % set initial conditions
for i = 2:length(t)              % for each time step
    x(i) = x(i-1) + v(i-1)*dt; % calculate position
    v(i) = v(i-1) + a(i-1)*dt; % calculate velocity
    a(i) = -1.5*sqrt(v(i));     % calcualte acceleration
end
subplot(3,1,1); plot(t,x); ylabel('x (m)'); xlabel('t (sec)')
title(sprintf('For a = -1.5*sqrt(v) and with dt = %g', dt))
text(0.1, 2.5, sprintf('x_0 = %d\nx_f_i_n_a_l = %g', x(1), x(end)))
subplot(3,1,2); plot(t,v); ylabel('v (m/s)'); xlabel('t (sec)')
text(0.1, 1.5, sprintf('v_0 = %d\nv_f_i_n_a_l = %g', v(1), v(end)))
subplot(3,1,3); plot(t,a); ylabel('a (m/s^2)'); xlabel('t (sec)')
text(0.1, -1.5, sprintf('a_0 = %d\na_f_i_n_a_l = %g', a(1), a(end)))
set(gcf, 'color', 'white'); toc
```



Problem #3

A car starts from rest and moves along a straight line with an acceleration $a = 3x^{-1/3}$ in m/s^2. Determine the car's velocity and position at $t = 6$ s.

An expression that relates a, v, and t would be handy, but the one that exists has a second derivative which cannot be 'separated'. Instead, leave out time for now.

Use $a = v\dfrac{dv}{dx} = 3x^{-\frac{1}{3}}$ => Separate variables: $vdv = 3x^{-\frac{1}{3}}dx$

Integrate: $\int_0^v vdv = \int_0^x 3x^{-\frac{1}{3}}dx$ => $\frac{1}{2}v^2 = \frac{3}{2}3x^{\frac{2}{3}} \Rightarrow v = 3x^{\frac{1}{3}}$

Use $v = \dfrac{dx}{dt} = 3x^{\frac{1}{3}}$ => Separate variables: $x^{-\frac{1}{3}}dx = 3dt$

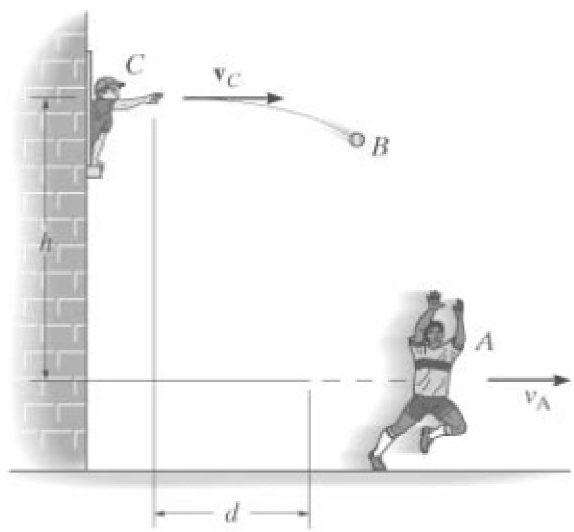
Integrate: $3\int_0^t dt = \int_0^x x^{-\frac{1}{3}}dx$

$$3t = \tfrac{3}{2}x^{\frac{2}{3}} \Rightarrow (2t)^{\frac{3}{2}} = x = (12)^{\frac{3}{2}} \boxed{= 41.5692 \text{ m}}$$

$$v = \tfrac{d}{dt}x = \tfrac{d}{dt}(2t)^{\frac{3}{2}} = 2\tfrac{3}{2}(2t)^{\frac{1}{2}} \boxed{= 10.3923 \text{ m/s}}$$

Problem #4

A is moving in a straight line away from the building at a constant speed 4 ft/s. C is at $h = 20$ ft and throws the ball B horizontally when A is at $d = 10$ ft. At what speed must C throw the ball so that A can catch it?



time for ball to drop $h = 20$ ft:

$y_B = (y_B)_0 + (v_B)_{y0} + ½(a_B)_yt^2$

$0 = 20 \text{ ft} + 0t - ½\, gt^2$

$$t = \sqrt{\frac{2 \cdot 20}{32.2}} = 1.114556 \text{ s}$$

$x_A = (x_A)_0 + (v_A)_{x0}t + ½(a_A)_xt^2$

$x_A = 10 \text{ ft} + (4 \text{ ft/s})(1.114556 \text{ s}) + ½0t^2 = 14.458 \text{ ft}$

$x_B = (x_B)_0 + (v_B)_{x0}t + ½(a_B)_xt^2$

$14.458 \text{ ft} = 0 + (v_B)_{x0}(1.114556 \text{ s}) + ½0t^2$ => $(v_B)_{x0} = (14.458 \text{ ft})/ (1.114556 \text{ s})$

$(v_B)_{x0} = 12.97$ ft/s